Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 5, 1996

CONSTRUCTION OF A NONLOCAL MODEL FOR DIFFERENT-MODULUS VISCOELASTIC MEDIA

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UDC 534.22

A nonlocal mathematical model for a different-modulus viscoelastic body is constructed using the Godunov–Romenskii model for a viscoelastic body and the technique of constructing different-modulus media developed by V. P. Myasnikov et al. This approach made it possible to describe, within a unique model, the rheological behavior of liquid and solid bodies and also transitions from an elastic state to plastic, semi-brittle, brittle, and completely fractured states of bulk media.

1. Model for a Different-Modulus Elastic Body. In Euclidian space the elastic-strain tensor is the Almansi strain tensor ($\varepsilon_{\alpha\beta}$ are the tensor components in the Cartesian coordinate system $x_\alpha, \alpha = \overline{1,3}$).

Let us consider the expansion of the density of internal energy E in a power series of $\varepsilon_{\alpha\beta}$ and entropy s for relatively small deformations. In the case of an isotropic body, E depends only on the strain-tensor invariants $I_i(\|\varepsilon_{\alpha\beta}\|)$, $i = \overline{1,3}$. Then,

$$\Phi = \overset{\circ}{\Phi} + \Phi'_s(s - \overset{\circ}{s}) + \frac{1}{2}\Phi''_{ss}(s - \overset{\circ}{s})^2 + \Phi'_{I_1} I_1 + \Phi''_{sI_1}(s - \overset{\circ}{s}) I_1 + \frac{1}{2}\Phi''_{I_1 I_1} I_1^2 + \frac{1}{2}\Phi'_{I_2} I_2 + \ldots \tag{1.1}$$

($\Phi = \overset{\circ}{\rho} E$, where $\overset{\circ}{\rho}$ is the initial density). The initial state is chosen so that elastic stresses are equal to zero ($p_{\alpha\beta} = 0$) for $\varepsilon_{\alpha\beta} = 0$, and then $\Phi'_{\varepsilon_{\alpha\beta}} = 0$. Hence, with accuracy to the additive constant $\overset{\circ}{\Phi}$ we have

$$\Phi = \overset{\circ}{\rho}\overset{\circ}{T}(s - \overset{\circ}{s}) + \frac{b_1}{2}(s - \overset{\circ}{s})^2 - b_2(s - \overset{\circ}{s}) I_1 + \frac{\lambda}{2} I_1^2 + \mu\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha} + \ldots,$$

where $b_1 = \Phi''_{ss}$; $b_2 = -\Phi''_{sI_1}$; $\lambda = \Phi''_{I_1 I_1} + (1/2)\Phi'_{I_2}$; $\mu = -(1/4)\Phi'_{I_2}$; $\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha} = I_1^2 - 2I_2$; and λ and μ are the Lamé parameters. Summation is always performed over repeating indices. Below, the strain and stress tensors are assumed to be symmetrical. In this case, $T - \overset{\circ}{T} = (b_1(s - \overset{\circ}{s}) - b_2 I_1)/\overset{\circ}{\rho}$ and $s - \overset{\circ}{s} = (\overset{\circ}{\rho}(T - \overset{\circ}{T}) + b_2 I_1)/b_1$ ($E'_s = T$ is the temperature). From the continuity equation with accuracy to second-order terms with respect to $\varepsilon_{\alpha\beta}$ we obtain $\rho = \overset{\circ}{\rho}\sqrt{\det\|\delta_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\|} = \overset{\circ}{\rho}(1 - I_1 + \ldots)$ ($\delta_{\alpha\beta}$ is the Kronecker symbol).

Elastic stresses are defined by the Murnaghan formulas [1]

$$\begin{aligned} p_{\alpha\beta} &= \rho(\delta_{\alpha k} - 2\varepsilon_{\alpha k})E'_{\varepsilon_{k\beta}} = (1 - I_1 + \ldots)(\delta_{\alpha k} - 2\varepsilon_{\alpha k})\Phi'_{\varepsilon_{k\beta}} \\ &\approx \lambda I_1\delta_{\alpha\beta} + 2\mu\varepsilon_{\alpha\beta} - b_2(s - \overset{\circ}{s})((1 - I_1)\delta_{\alpha\beta} - 2\varepsilon_{\alpha\beta}) \\ &= \lambda I_1\delta_{\alpha\beta} + 2\mu\varepsilon_{\alpha\beta} - \frac{b_2^2}{b_1} I_1\delta_{\alpha\beta} - \frac{b_2}{b_1}\overset{\circ}{\rho}(T - \overset{\circ}{T})((1 - I_1)\delta_{\alpha\beta} - 2\varepsilon_{\alpha\beta}). \end{aligned} \tag{1.2}$$

Hooke's law is extended to the case where temperature stresses and strains are taken into account.

Different approaches are applied to the construction of models for irregular media. For microinhomogeneous media, effective moduli of elasticity of various types are usually introduced. We use the technique developed in [2–4].

Computing Center, Far East Division, Russian Academy of Sciences, Khabarovsk 680063. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 5, pp. 154–161, September–October, 1996. Original article submitted January 4, 1995; revision submitted June 13, 1995.

Let us consider an elastic potential of the form

$$U = \frac{1}{2}(L + K(\xi))I_1^2 + [M + G(\xi)]E_2, \tag{1.3}$$

where $\xi = I_1/(\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha})^{1/2}$; $E_2 = \varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha} - I_1^2/3$; $L = \lambda + 2\mu/3$; and $M = \mu$. The functions $K(\xi)$ and $G(\xi)$ are additive corrections to the bulk modulus and the shear modulus of the Hooke's law that are caused by the sensitivity of moduli to variation of ξ.

From the condition $U''_{I_1E_2} = U''_{E_2I_1}$ we obtain restrictions on the selection of corrections:

$$3\xi^2 K'_\xi + 2(3 - \xi^2)G'_\xi = 0. \tag{1.4}$$

In this case,

$$p_{\alpha\beta} = (L + K(\xi))d\delta_{\alpha\beta} + 2(M + G(\xi))d_{\alpha\beta}, \tag{1.5}$$

where $d = I_1 = \varepsilon_{\alpha\alpha}$ and $d_{\alpha\beta} = \varepsilon_{\alpha\beta} - d\delta_{\alpha\beta}/3$ are the components of the strain deviator. It follows from (1.5) that the mean stress $p = p_{\alpha\alpha}/3$ and the stress intensity $p_0 = \sqrt{3s_{\alpha\beta}s_{\beta\alpha}/2}$ ($s_{\alpha\beta} = p_{\alpha\beta} - p\delta_{\alpha\beta}$ is the stress deviator) depend on strains:

$$p = (L + K(\xi))d = \Omega(\xi)d, p_0 = 3(M + G(\xi))d_0 = 3W(\xi)d_0. \tag{1.6}$$

Here $d_0 = \sqrt{2d_{\alpha\beta}d_{\beta\alpha}/3} = \sqrt{2E_2/3}$ and $W(\xi) \geqslant 0$. The volume and shear strains are connected by the relation $I_1 = (\xi/\sqrt{1 - \xi^2/3})\sqrt{E_2}$ from which $|\xi| \leqslant \sqrt{3}$.

We define the functions $K(\xi)$ and $G(\xi)$. For this purpose, according to (1.4) and (1.6), it is sufficient to find only one of them. If $K(\xi) = \Omega(\xi) - L$, then

$$G(\xi) = \frac{3}{2}\int_0^\xi \frac{\xi^2}{\xi^2 - 3}\Omega'_\xi(\xi)\, d\xi + G(0).$$

One can set $G(0) = 0$. The constant $M = W(0)$ is determined for pure shear deformation. If one uses the relation $G(\xi) = W(\xi) - M$, then

$$K(\xi) = \frac{2}{3}\int_{-\sqrt{3}}^{\xi} \frac{\xi^2 - 3}{\xi^2}W'_\xi(\xi)\, d\xi + K(-\sqrt{3}).$$

The constant L is determined from an experiment on hydrostatic compression when $\xi = -\sqrt{3}$, $L = \Omega(-\sqrt{3}) - K(-\sqrt{3})$.

Let us find the dependence of strains on stresses. We introduce the variable

$$\gamma = \frac{p}{p_0} = \frac{\xi(L + K(\xi))}{\sqrt{2(3 - \xi^2)}(M + G(\xi))}. \tag{1.7}$$

Solving (1.7) for ξ, we obtain $\xi = \xi(\gamma)$. From (1.6) we have

$$d = f_1(\gamma)p, \quad f_1(\gamma) = [L + K(\xi(\gamma))]^{-1}, \quad d_{\alpha\beta} = \frac{1}{2}f_2(\gamma)s_{\alpha\beta}, \quad f_2(\gamma) = [M + G(\xi(\gamma))]^{-1}.$$

Hence, we obtain the required dependence

$$\varepsilon_{\alpha\beta} = \frac{1}{3}f_1(\gamma)p\delta_{\alpha\beta} + \frac{1}{2}f_2(\gamma)s_{\alpha\beta}. \tag{1.8}$$

Oleinikov [4] has shown the agreement of the different-modulus model and the experimental data which were derived in a series of tests on proportional loading of rocks. He has concluded that the function $G(\xi)$ for most tested rocks (diabase, coal, limestone, cement, and salt) can be satisfactorily approximated by a linear function and is decreasing. Using this, we set $G(\xi) = -\nu\xi/2$, $\nu \geqslant 0$ is const. Then,

$$K(\xi) = -\frac{\nu}{3}\Big(\xi + \frac{3}{\xi} + 2\sqrt{3}\Big) + K(-\sqrt{3}) = -\frac{1}{3}\nu\Big(\xi + \frac{3}{\xi}\Big).$$

Here without loss of generality one can assume $K(-\sqrt{3}) = 2\sqrt{3}\nu/3$. In this case,

$$U = \frac{\lambda}{2}I_1^2 + \mu\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha} - \nu I_1\sqrt{\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha}} = \frac{\lambda^e}{2}I_1^2 + \mu^e\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha},$$

$$p_{\alpha\beta} = (\lambda I_1 - \nu\sqrt{I_1^2 - 2I_2})\delta_{\alpha\beta} + \Big(2\mu - \nu\frac{I_1}{\sqrt{I_1^2 - 2I_2}}\Big)\varepsilon_{\alpha\beta} = \lambda^e I_1\delta_{\alpha\beta} + 2\mu^e\varepsilon_{\alpha\beta}, \tag{1.9}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2(M + G(\xi))}\Big[p_{\alpha\beta} + \Big(\frac{2}{3}\frac{M + G(\xi)}{L + K(\xi)} - 1\Big)p\delta_{\alpha\beta}\Big] = \frac{1}{2\mu^e}\Big(p_{\alpha\beta} - \frac{\lambda^e}{\lambda^e + 2\mu^e/3}p\delta_{\alpha\beta}\Big),$$

where $\lambda^e = \lambda - \nu/\xi$ and $\mu^e = \mu - \nu\xi/2$ are the effective elastic moduli; the value of ξ in the relations for strains is determined in terms of γ from (1.7). The elastic potential includes an additional term that enables one to allow for the dependence of the elastic moduli on the type of stress state and their stepwise change in transition from tension to compression.

Using the thermodynamic requirement of convexity of the elastic potential (of positive definiteness of the quadratic form of U), we obtain additional restrictions on the choice of model parameters. Taking into account the relations for the strain-tensor deviator $\varepsilon_{\alpha\beta} - (1/3)I_1\delta_{\alpha\beta} = (1/2\mu^e)(p_{\alpha\beta} - p\delta_{\alpha\beta})$, from (1.6) and (1.9) we have

$$U = \frac{1}{2(\lambda^e + 2\mu^e/3)}p^2 + \frac{1}{4\mu^e}(p_{\alpha\beta} - p\delta_{\alpha\beta})(p_{\alpha\beta} - p\delta_{\alpha\beta}).$$

The elastic potential is positively defined when

$$2\mu - \nu\xi > 0, \qquad 3\lambda + 2\mu - \nu(\xi + 3/\xi) > 0. \tag{1.10}$$

Let us use the representation (1.9) for U. From positive definiteness of the major minors of the matrix

$$\begin{Vmatrix} \mu & -\nu/2 \\ -\nu/2 & \lambda/2 \end{Vmatrix}$$

we obtain the missing restriction

$$\nu^2 < 2\mu\lambda. \tag{1.11}$$

Relation (1.11) gives the limiting value of $\nu^2 = 2\mu\lambda$, for which, when $I_1 > 0$, the potential loses convexity, and relation (1.10) $\nu = 0$ gives the convexity conditions for an elastic (Hooke's) medium for $\nu = 0$.

Thus, the model for a different-modulus elastic body with a linearly decreasing dependence of the shear modulus on the parameter ξ is completely described. Construction of models for different-modulus media enables one to take into account the behavior of media relative to the sign and type of loading, which is due to material microinhomogeneity.

2. General Equations of the Macroscopic Model for a Different-Modulus Viscoelastic Medium. For strongly irregular media, one should apply a phenomenological approach. The basic principles of nonlocal models are developed in [5]. In this paper, we use a model for a viscoelastic body [1] in which the elastic parameters depend on the fracturing of the medium.

The equations of the laws of conservation of mass, momentum, and energy for multicomponent media in the Cartesian coordinate system are as follows:

$$\frac{\partial\rho}{\partial t} + \nabla\rho\mathbf{v} = 0, \qquad \rho\frac{dc_i}{dt} = -\nabla\mathbf{J}_i + M_i\sum_{j=1}^{n}\nu_{ij}l_j \quad (i = \overline{1, N-1}),$$
$$\rho\frac{dv_\alpha}{dt} = \frac{\partial}{\partial x_\beta}P_{\alpha\beta} + \rho F_\alpha \quad (\alpha = \overline{1,3}), \qquad \rho\frac{dE}{dt} = P_{\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta} + \rho\frac{dq^e}{dt} + \rho\frac{dq^*}{dt}, \tag{2.1}$$

where t is time; $\rho = \sum_{i=1}^{N} \rho_i$ is the density; $\mathbf{v} = \sum_{i=1}^{N} c_i \mathbf{v}_i$ is the velocity; M_i, ρ_i, $c_i = \rho_i/\rho$, $\mathbf{v}_i$, and $\mathbf{J}_i = \rho_i(\mathbf{v}_i - \mathbf{v})$ are the molecular weight, density, mass concentration, velocity, and diffusion flow of the ith component; l_j and ν_{ij} are the velocity and stoichiometric coefficients of the jth chemical energy ($\sum_{i=1}^{N} \nu_{ij} - 0$); $\|P_{\alpha\beta}\|$, $\mathbf{F}$, dq^e, and dq^* are the stress tensor, external mass force, heat influx, and nonthermal energy sources; and d/dt is the substantial derivative.

Elastic parameters depend on fracturing of the medium. In the same way as in the models of [6, 7], we introduce the parameter u for the degree of material damage $0 \leqslant u \leqslant 1$. When $u = 0$, there are no microcracks. Elastic strains are described by Hooke's model ($\nu = 0$). When $u = 1$, the medium is completely fractured ($\nu^2 = 2\mu\lambda$). Intermediate states ($0 < u < 1$) are described by the model of different-modulus media.

The u variation results in energy dissipation. Let us write the balance equations of energy and entropy:

$$dE = E'_s\, ds + E'_{\varepsilon_{\alpha\beta}} d\varepsilon_{\alpha\beta} + E'_{c_i} dc_i + E'_u du + E'_{\nabla_\alpha u} d\nabla_\alpha u, \qquad T ds = dq^e + dq' \tag{2.2}$$

(dq' is the uncompensated heat). Simplifying the last term in (2.2)

$$E'_{\nabla_\alpha u} \frac{d\nabla_\alpha u}{dt} = \nabla_\alpha \Big(E'_{\nabla_\alpha u} \frac{du}{dt} \Big) - (\nabla_\alpha E'_{\nabla_\alpha u}) \frac{du}{dt} - E'_{\nabla_\alpha u} \nabla_\beta u \nabla_\alpha v_\beta,$$

we represent the stress tensor by the sum of elastic, viscous, and structural stresses:

$$P_{\alpha\beta} = p_{\alpha\beta} + \sigma_{\alpha\beta} - E'_{\nabla_\alpha u} \nabla_\beta u. \tag{2.3}$$

Then, from (2.1) and (2.2) we obtain

$$\frac{dq^*}{dt} = J + \nabla_\alpha \Big(E'_{\nabla_\alpha u} \frac{du}{dt}\Big),$$

$$\frac{dq'}{dt} = \frac{1}{\rho}(p_{\alpha\beta} + \sigma_{\alpha\beta}) \nabla_\beta v_\alpha - E'_{\varepsilon_{\alpha\beta}} \frac{d\varepsilon_{\alpha\beta}}{dt} - E'_{c_i} \frac{dc_i}{dt} - \frac{\delta E}{\delta u} \frac{du}{dt} + J.$$

Here J is the intensity of nonthermal sources (for example, radioactive ones); $\delta E/\delta u = E'_u - \nabla_\alpha E'_{\nabla_\alpha u}$ is the variational derivative.

Elastic stresses are defined by the Murnaghan formulas (1.2), and the elastic-stress tensor satisfies the equation

$$\frac{d\varepsilon_{\alpha\beta}}{dt} = \frac{1}{2}\Big(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\Big) - \varepsilon_{\alpha k} \frac{\partial v_k}{\partial x_\beta} - \varepsilon_{k\beta} \frac{\partial v_k}{\partial x_\alpha} + \frac{1}{2}\varphi_{\alpha\beta}$$

($\|\varphi_{\alpha\beta}\|$ is the relaxation matrix). Then,

$$p_{\alpha\beta} \frac{\partial v_\alpha}{\partial x_\beta} = \rho E'_{\varepsilon_{\alpha\beta}} \Big(\frac{d\varepsilon_{\alpha\beta}}{dt} - \frac{1}{2}\varphi_{\alpha\beta} \Big). \tag{2.4}$$

To describe the rheological properties of the medium, we shall use the linear phenomenological relations

$$\begin{gathered} p_{\alpha\beta} = L^e I_1 \delta_{\alpha\beta} + 2M^e d_{\alpha\beta} - \frac{b_2^2}{b_1} I_1 \delta_{\alpha\beta} - \frac{b_2}{b_1} \mathring{\rho}\, (T - \mathring{T})((1 - I_1)\delta_{\alpha\beta} - 2\varepsilon_{\alpha\beta}), \\ \sigma_{\alpha\beta} = (\zeta(\nabla \mathbf{v}) - H)\delta_{\alpha\beta} + 2\eta e_{\alpha\beta} = \sigma'_{\alpha\beta} - H\delta_{\alpha\beta}, \end{gathered} \tag{2.5}$$

where $L^e = \lambda^e + 2\mu^e/3$; $M^e = \mu^e$; η and ζ are the shear and volume viscosities; and H is a scalar function of $(\nabla \mathbf{v})$ that is nonlinear in a general case;

$$e_{\alpha\beta} = \frac{1}{2}\Big(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}(\nabla \mathbf{v}) \Big).$$

We introduce the heat flux vector $\mathbf{J}_q$ to reduce the heat-influx dq^e to the divergent form $dq^e = -(1/\rho)(\nabla \mathbf{J}_q)dt$. Using (2.4) and the equations of conservation of the concentration of the components, we write the equation of entropy balance as

$$\rho\frac{ds}{dt} = \nabla \mathbf{J}_s + \sigma_s, \qquad \mathbf{J}_s = -\frac{1}{T}(\mathbf{J}_q - \sum_{i=1}^{N} \mu_i \mathbf{J}_i),$$

$$\sigma_s = \frac{\rho J}{T} + \frac{\sigma_{\alpha\beta}}{T}\frac{\partial v_\alpha}{\partial x_\beta} - \frac{1}{T}\sum_{j=1}^{n} A_j l_j - \frac{1}{T^2}(\nabla T\cdot \mathbf{J}_q) - \sum_{i=1}^{N}(\mathbf{J}_i\cdot\nabla\frac{\mu_i}{T}) - \frac{\rho}{2T}E'_{\varepsilon_{\alpha\beta}}\varphi_{\alpha\beta} - \frac{\rho}{T}\frac{\delta E}{\delta u}\frac{du}{dt}.$$

Here $\mathbf{J}_s$ is the entropy flux; σ_s is the power of energy sources; $A_j = \sum_{i=1}^{N} \mu_i M_i \nu_{ij}$; and $\mu_i = E'_{c_i}$ are the chemical potentials of the components.

According to the second thermodynamics law, $\sigma_s \geqslant 0$ and the equality is achieved only for reversible processes. The condition of nonnegativity of internal entropy production imposes additional restrictions on the relationships between thermodynamical forces and fluxes. Let us formulate these restrictions.

By virtue of the Curie symmetry principle [8],

$$\frac{1}{T}\sigma'_{\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta} - \frac{\rho}{2T}E'_{\varepsilon_{\alpha\beta}}\varphi_{\alpha\beta} \geqslant 0, \qquad -\frac{1}{T^2}(\nabla T\cdot\mathbf{J}_q) - \sum_{i=1}^{N}\Big(\mathbf{J}_i\cdot\nabla\frac{\mu_i}{T}\Big) \geqslant 0,$$
$$-\frac{1}{T}\sum_{j=1}^{n} A_j l_j - \frac{1}{T}H(\nabla\cdot \mathbf{v}) \geqslant 0, \qquad -\frac{\rho}{T}\frac{\delta E}{\delta u}\frac{du}{dt} \geqslant 0. \tag{2.6}$$

Using the dissipativity postulate $\sigma'_{\alpha\beta}\nabla_\beta v_\alpha \geqslant 0$, which holds when $\eta \geqslant 0$ and $\zeta \geqslant 0$, we have $-(\rho/T)E'_{\varepsilon_{\alpha\beta}}\varphi_{\alpha\beta} \geqslant 0$. Relaxation and friction are irreversible dissipative processes, each increasing entropy. Relaxation processes can be specified by the relations [9, 10] $\varphi_{\alpha\beta} = -(2/\tau)(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta}\rho'_{\varepsilon_{lk}}\varepsilon_{lk}/\rho'_{\varepsilon_{ll}})$, where τ is the relaxation time.

To describe thermodynamic fluxes, we use linear phenomenological relations with symmetrical coefficients. Then, the second inequality in (2.6) is valid in the case of positive definiteness of the matrix of phenomenological coefficients [9]. Heat and diffusion fluxes are usually written as

$$\mathbf{J}_i = -\rho\sum_{l=1}^{N-1} D_{il}\nabla c_l - \rho S_i \nabla\ln p - \rho D_{Ti}\nabla\ln T,$$

$$\mathbf{J}_q = -æ\nabla T + \sum_{i=1}^{N} h_i \mathbf{J}_i - \rho\sum_{i=1}^{N-1} D_{Ti}(\frac{\partial(\mu_i-\mu_N)}{\partial p}\nabla p + \sum_{l=1}^{N}\frac{\partial(\mu_i-\mu_N)}{\partial c_l}\nabla c_l).$$

Here $h_i = -T^2\partial(\mu_i/T)/\partial T$ is the enthalpy of the ith component of the mixture; D_{il}, S_i, and D_{Ti} are the diffusion, barodiffusion (sedimentation), and thermal diffusion coefficients; æ is the thermal conductivity coefficient; and p is the pressure.

To describe chemical reactions, we use the general thermodynamical relations for nonequilibrium processes. In phenomenological description, the number n of independent reactions cannot exceed N. We assume that $n = N$. The general relations are of the form

$$H = -\frac{\partial K}{\partial(\nabla\cdot\mathbf{v})}, \qquad l_j = -\frac{\partial K}{\partial A_j}, \tag{2.7}$$

where $K(p, T, A_1 \dots A_n, \nabla \mathbf{v})$ is the dissipative chemical potential which describes the chemical properties of mixtures. For them the required inequality should be satisfied.

From the viewpoint of thermodynamics of nonequilibrium processes, physicochemical transformations are a particular case of possible scalar relaxation processes. Analysis of their interaction with the viscous-flow processes shows [11] that the rheological effect of chemical reactions is associated with volume viscosity (2.5).

Volume viscosity changes significantly when the dependence of l_j on the rate of volume change of a mixture component is strong. For most rocks viscous stresses are primarily determined by shear viscosity. In this case, volume viscosity and the dependence of the dissipative chemical potential on $\nabla \mathbf{v}$ can be ignored:

$$\sigma_{\alpha\beta} = 2\eta e_{\alpha\beta}.$$

From (2.6) we obtain fracture kinetics:

$$\frac{du}{dt} = -C\frac{\delta E}{\delta u}. \tag{2.8}$$

The elastic parameters λ, μ, and ν depend on u. For example,

$$\lambda = \lambda_0(1-f) + \lambda_1 f, \qquad \mu = \mu_0(1-f) + \mu_1 f, \qquad \nu = \nu_1 f. \tag{2.9}$$

Here λ_0 and μ_0 are the parameters for $u = 0$; λ_1, μ_1, and $\nu_1 = 2\mu_1\lambda_1$ are the parameters for $u = 1$, $f(0) = 0$, and $f(1) = 1$. As a function $f(u)$, one can use the volume portion of the damaged material of the element under consideration. Then, the medium can be considered a mechanical mixture of cured and fractured materials.

Substituting (2.9) into (1.9), we obtain

$$U = (1-f)U_0 + fU_1, \tag{2.10}$$

where $U_0 = (\lambda_0/2)I_1^2 + \mu_0\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha}$ and $U_1 = (\lambda_1/2)I_1^2 + \mu_1\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha} - \nu_1 I_1\sqrt{\varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha}}$.
The dependence of E on u and ∇u is specified in the form

$$E(u, \nabla u) = D(u-u_1)^2(u-u_2)^2 + \delta\nabla_\alpha u\nabla_\alpha u \tag{2.11}$$

(u_1 and u_2 are degrees of fracture that correspond to the given temperature and zero load in transient and brittle states of medium). Using the fourth-order polynomial in u enables one to describe the existence of two stable steady states to one of which (depending on initial conditions) the medium relaxes upon load relief. These two steady states of equilibrium are separated by the intermediate unsteady state $u_3 = (u_1 + u_2)/2$.

In a certain range of variation of stress and temperature the medium can be in a state of relatively stable physical properties (the values of physical quantities are close to those at equilibrium), and then it changes its properties, jumpwise entering into another rheological state. Thus, the degree of fracture is close to the corresponding values of (0, u_1, u_2, and 1). With change in the rheological state, the degree of fracture changes jumpwise.

It is necessary to ensure the continuity of transition of bulk media ($u = 1$ and $2\mu\lambda = \nu^2$) to a limiting unsteady state in which when the elastic potential loses convexity: $U = 0$ when $\xi = \xi^* = 2\mu/\nu$.

The limiting state is achieved when the loading point emerges on the fracture surface, i.e., at loads resulting in discontinuity decay or shear that can grow without an increase in the load. To describe the behavior of bulk media and the features of deformation in the limiting state, one can use a deformation model. The model takes into account the governing properties of the media such as dilatancy, zero resistance to tensile stresses, limited ability to strengthening, and the instability of limiting deformation. The equations that relate strains with stresses under loading of bulk media in terms of deformation theory are constructed in [12, 13].

Considering the properties of the model, one can assume that the viscosity coefficients are variable parameters. The state of a material that is stable with respect to the degree of fracture is consistent with the regime of steady-state creep (accumulation of irreversible strains). A change in the degree of fracture leads to a change in the creep rate and a jump of viscosity. At a low degree of fracture and $-\sqrt{3} \leqslant \xi \leqslant \xi_0$ (overall compression and small shear), a high-temperature creep regime can occur. At high temperatures, viscosity decreases sharply. To describe such transitions, it suffices to assume the presence of jumps in the dependence of the medium's viscosity on u, T, and p.

The quadratic dependence of energy on ∇u is the simplest for an isotropic medium. In accordance with (2.10) and (2.11), the stress tensor and the equation of kinetics of u become

$$P_{\alpha\beta} = \lambda^e I_1\delta_{\alpha\beta} + 2\mu^e\varepsilon_{\alpha\beta} + (\zeta(\nabla\mathrm{v}) - H)\delta_{\alpha\beta} + 2\eta e_{\alpha\beta} - 2\delta\nabla_\alpha u\nabla_\beta u,$$

$$\frac{du}{dt} = C\Big(\frac{1}{\overset{\circ}{\rho}}f'_u(U_0 - U_1) - 4D(u-u_1)(u-u_2)\Big(u - \frac{u_1+u_2}{2}\Big) + 2\delta\Delta u\Big),$$

where $\Delta u = \nabla_\alpha\nabla_\alpha u$.

The equation of kinetics of u can be used to describe the dynamic effects associated with transition from one rheological state to another. The transition zone or the failure front are determined by specifying the parameter δ.

The values of the parameters λ_0, μ_0, λ_1, μ_1, D, u_1, u_2, and δ should be determined experimentally, and the selection of the initial distributions of u and $f(u)$ should be consistent with the properties of the quantities obtained from qualitative analysis.

Thus, we constructed a model for a different-modulus viscoelastic body that makes it possible to describe the presence of various rheological states and to take into account the different-modulus properties of material and the nonlocal character of failure. The model is sufficiently simple and can be used for numerical simulation of complicated processes in irregular media, and for solution of a number of practical problems in geophysics and mining engineering.

The work was supported by the Russian Foundation for Fundamental Research (Grant 94-01-00743).

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